Homework 1 : Surface Interrogation using Curvatures

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1 Problem Specification

The homework is to explore surface interrogation with the help of various kinds of curvatures. According to Patrikalakis and Maekawa[1], "Interrogation is the process of extraction of information from a geometric model." There are numerous ways to extract information from a geometric models. We will be trying to do the same using surface curvatures. These come under the second-order interrogation methods as classified by[1]. The task is to use the Gaussian, Mean, RMS and Absolute curvatures[2 3 4 5] for surface interrogation.

After the homework, we should be able to tell the effectiveness of each curvatures in evaluating the shapes. We should be able to have an idea about which kind of curvature is best suited for which kind of surfaces. Thus, we need to explore the strengths and weaknesses of each curvature measures, specific to various kind of surfaces.

2 Curvatures

Before getting into the nuances of shape interrogation using curvatures, let us know what do we mean by curvatures. A curvature is a generic term used to describe the difference in properties of an geometric object compared to a standard set of objects which are considered to be flat[6]. It comprises of various measurable characteristic which can be used to tell two objects apart, in our case a curved surface from a plane. Let us understand the curvature for a surface in detail.

2.1 Surface Curvature

If we take a two-dimensional surface in a 3-dimensional space, and draw a one-dimensional curve on it, we can measure the curvatures of the curve using various curvature measures. The three main types of curvature that we can see in literature are the normal curvature(κ_n), geodesic curvature(κ_g) and geodesic torsion. Given any point on a surface, there will be a tangent vector $\mathbf{T}$ on that

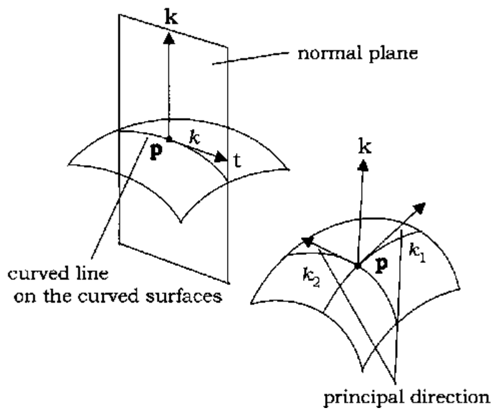


Figure 1: Curvature on a Surface

point defined by the surface. This tangent vector will lie on a tangent plane of the surface orthogonal to the normal vector as shown in Figure 1.

The curvature of the curve projected on the plane containing the curve, the tangent vector and the normal vector is known as the normal curvature(κ_n) of the surface at that point[7]. The plane containing the vector is termed as the normal plane. The geodesic curvature(κ_g) is defined as the curvature of the curve projected on the tangent plane.

If we rotate the normal plane about the normal at the given point, we obtain different value of curvatures, depending on the shape of the surface. The minimum(κ_{min}/κ_1) and the maximum(κ_{max}/κ_2) value of normal curvature on a surface is termed as the principal curvature of the surface. These principal curvatures are the eigen values of the differential of the Gauss map (∂N).

2.2 Gaussian Curvature

The Gaussian curvature of a surface at any point is simple the product of their principle curvatures[8] found using the normal curvature at the point on the surface.

$$\kappa_{gauss}(K) = \kappa_{max} * \kappa_{min}$$

In context of Gaussian curvature, it maybe useful to know three types of surface and how related to the Gaussian curvature, viz. Developable surface, sphere, and pseudospherical surface.

A developable surface has constant zero Gaussian curvature value and the geometry of the surface is in euclidean space. The surfaces having a constant positive Gaussian curvature is a sphere and follows spherical geometry. If the surface has a constant negative Gaussian curvature, then it is a pseudospherical surface and follows the hyperbolic geometry.

It is very interesting to relate the sign of the Gaussian curvature with the principal curvatures as developable surface will always have zero principal curvature values, sphere will have a constant positive value for both the principal curvature and a psuedospherical shape will have a positive maximum principal curvature while a negative minimum principal curvature for the overall Gaussian curvature to be negative.

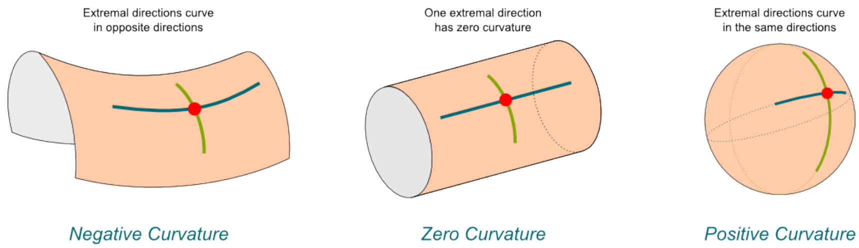


Figure 2: Gaussian Curvature

2.3 Mean Curvature

Using the principal curvatures obtained bu the normal curvature(κ_n), the mean curvature[9] of a surface at that given point is defined as,

$$\kappa_{mean}(H) = \frac{1}{2}(\kappa_{max} + \kappa_{min})$$

A surface having zero mean curvature is known as a minimal surface. Catenoid[10], Helicoid[11] and Enneper surface[12] are some classical minimal surfaces.

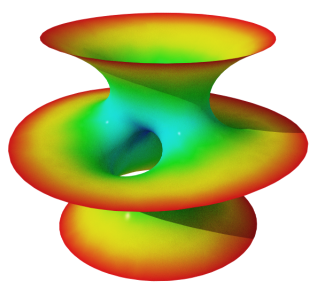


Figure 3: Costa's minimal surface, an example of zero mean curvature

2.4 RMS Curvature

The RMS or root mean square curvature is defined in terms of the prinicpal curvature as follows,

$$\kappa_{rms} = \sqrt{\kappa_{max}^2 + \kappa_{min}^2}$$

According to Maekawa[5], an absolute curvature is suitable for calculating the curvature of surfaces for meshing applications. But the absolute curvature is not a differentiable function. Hence, RMS curvatures are defined, which is differentiable and fairly useful for meshing purpose.

2.5 Absolute Curvature

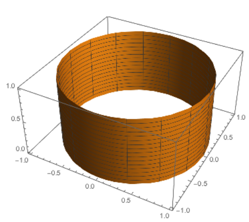
The absolute curvature[4] is defined as,

$$\kappa_{abs} = |\kappa_{max}| + |\kappa_{min}|$$

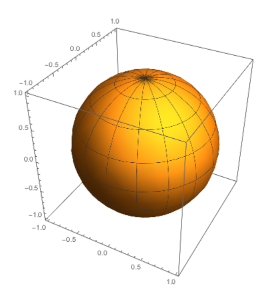
3 Surfaces and Shapes Used

In our experiments we have used various surfaces. There are essentially two types of exploration we have done in our experiments. The first part comprises of the surfaces whose geometry is well defined and has been studied extensively in the literature. These comprises of the basic shapes of a cylinder(Figure 4a), sphere(Figure 4b) and a torus(Figure 4c). For each of these shape curvature plots are made and compared with the standard results. These serve the purpose to check whether our method of implementation of these surface curvature are correct or not by comparing them with the standard results found in literature.

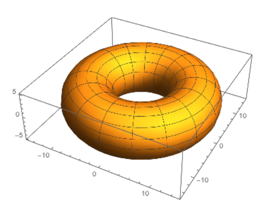
The second part comprises of the actual exploration. We create 7 parametric surface, each having a distinct shape and unique properties of their own. For ease of naming them, we have named them according to the curvature values of the curve, viz. generic graph(Figure 5a), positive upward curve(Figure 5b), positive downward curve(Figure 5c), negative curve(Figure 5d), zero curve(Figure 5e), monkey saddle(Figure 5f) and a modified four point saddle surface(Figure 5g). Having no idea how these surfaces will behave to different curvature measures, it was interesting to notice their behaviour and study them.



(a) Cylindrical Surface

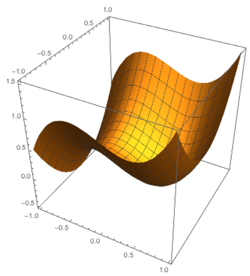


(b) Spherical Surface

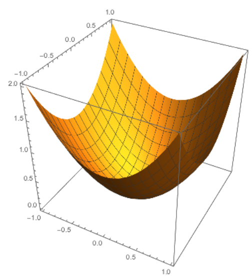


(c) Torus

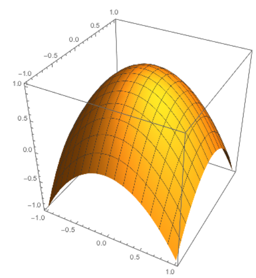
Figure 4: Different Surfaces for Shape Interrogation: Part I



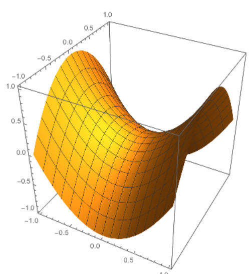
(a) Generic Graph



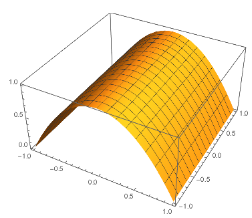
(b) Upward Positive Curvature Surface



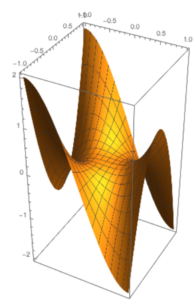
(c) Downward Positive Curvature Surface



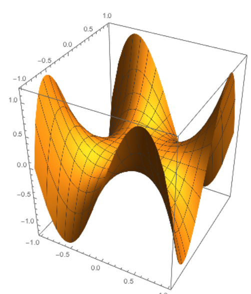
(d) Negative Curvature Surface



(e) Zero Curvature Surface



(f) Monkey Saddle



(g) Degenerate Four-Point Saddle

Figure 5: Different Surfaces for Shape Interrogation: Part II

4 Implementation

We can find the curvature of a surface using differential geometry on the surface. In Section 2, we saw how to find the Gaussian, Mean, RMS and Absolute curvatures with the help of Normal curvature. This is one way of calculating the curvatures. However, in this homework, we have used the First and Second fundamental forms to describe the surfaces and then define the shape shifters which are essentially used to calculate the Gaussian and Mean curvatures. Then we find the principal curvatures in terms of Gaussian and Mean curvatures using which we define the RMS and Absolute curvatures as described in Section 2.4 and 2.5.

The calculation of fundamental forms of a surface requires the calculation of partials in direction of the basis vectors.

The first fundamental form for any surface is given as G from the equation,

$$\partial l^2 = \partial u^T G \partial u$$

where G is given by

$$G = \begin{bmatrix} < x_1, x_1 > & < x_1, x_2 > \\ < x_2, x_1 > & < x_2, x_2 > \end{bmatrix}$$

We define, $E =< x_1, x_1 >, F =< x_1, x_2 >=< x_2, x_1 > andG =< x_2, x_2 >$, then first fundamental form for a surface is given by,

$$G(I) = eg - f^2$$

For the calculation fo second fundamental form, we take second derivative of the surface parameters along the basis vectors. The second fundamental is given by B where B is defined as,

$$B =< S(v), w >= w^T B w$$

where $S(v)$ are the Weingarten Shape operators. Here v is the parametrization coordinate. In terms of euclidean coordinates B is given as

$$B = \begin{bmatrix} < S(x_1), x_1 > & < S(x_1), x_2 > \\ < S(x_2), x_1 > & < S(x_2), x_2 > \end{bmatrix} = - \begin{bmatrix} < \partial N_{u1}, x_1 > & < \partial N_{u1}, x_2 > \\ < \partial N_{u2}, x_1 > & < \partial N_{u2}, x_2 > \end{bmatrix}$$

We define, $e =< \partial N_{u1}, x_1 >, f =< \partial N_{u1}, x_2 >=< \partial N_{u2}, x_1 > andg =< \partial N_{u2}, x_2 >$, then second fundamental form for a surface is given by,

$$B(II) = EG - F^2$$

In our experiments, we first calculate the partials for a surface, given a set of parameters. We then evaluate the partials and the curvatures for a range of values for the surface and then plot the values on the surface using a color map whose range is defined by the principal curvatures.

The Gaussian curvature in terms of the shape operators is given by,

$$K = det(\partial N) = \frac{eg - f^2}{EG - F^2} = \frac{B}{G} = \frac{IIndfundamentalform}{Istfundamentalform}$$

The Mean curvature in terms of the shape operators is given by,

$$H = \frac{1}{2}\frac{eG - 2fF - gE}{EG - F^2}$$

The principal curvatures in terms of the Gaussian Curvature (K) and Mean Curvature (H) are given by,

$$K_{max} = H + \sqrt{H^2 - K}$$

$$K_{min} = H - \sqrt{H^2 - K}$$

The RMS and Absolute Curvatures are given by the equation,

$$\kappa_{rms} = \sqrt{K_{max}^2 + K_{min}^2}$$

and

$$\kappa_{abs} = |K_{max}| + |K_{min}|$$

In the notebook we define functions to calculate all the partials and curvatures, and then plot the surfaces using *ParametricPlot3D* function defined in Mathematica, passing the parametric equation to the function and adding the temperature color map normalized to the range of principal curvatures and a legend to the properties of the plot.

5 Observations

The Curvature values and their respective minimum and maximum values for part I is given in the Table 1 and for part II is given in Table 2

The plots for each curve with their temperature color map as observed along with legends are as shown.

For Part 1, on the cylindrical surface, all four curvatures were calculated. The plots are given in Figure 6. For the spherical and torus curves, the Gaussian, Mean and absolute curves were calculated and they are shown in the Figure 7 and Figure 8 respectively.

For Part 2, on the generic graph, positive upward curve and positive downward curve the RMS value were not calculated. The plots are given in Figure 9, Figure 10 and Figure11 respectively. For all other surfaces, all four curvatures were calculated and they are shown in the Figure 12, Figure 13, 14, Figure 15.

Shapes	Gaussian			Mean		
	κ_1	κ_2	K	κ_1	κ_2	H
Cylinder	0	0	0	0.5	0.5	0.5
Sphere	1	1	1	$-1.336x10^{-16}$	$1.116x10^{-16}$	$-0.11x10^{-16}$
Torus	0.0085	0.0133	$1.1305x10^{-4}$	0.067	0.079	0.073

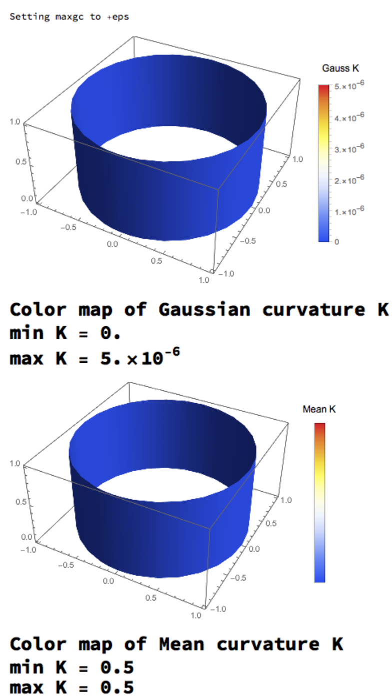
Shapes	RMS			Absolute		
	κ_1	κ_2	κ_{RMS}	κ_1	κ_2	κ_{Abs}
Cylinder	1	1	$\sqrt{2}$	1	1	2
Sphere	-	-	-	2	2	4
Torus	-	-	-	0.184	0.231	0.415

Table 1: Part I - Evaluating Curvatures on known Surfaces

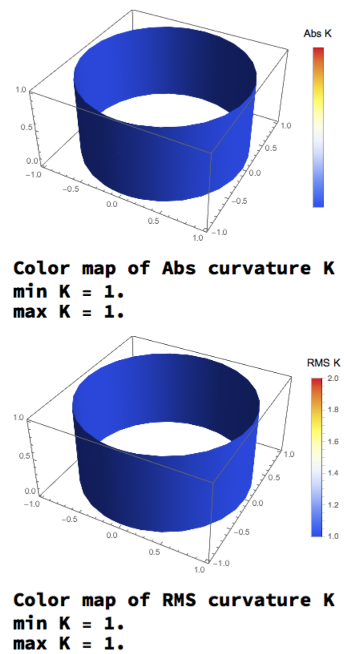
Shapes	Gaussian		Mean	
	κ_{min}	κ_{max}	κ_{min}	κ_{max}
Generic Graph	0.095	5.361	-0.894	0.679
Positive Upward Curve	0.049	4	-0.385	0.385
Positive Downward Curve	0.049	4	-0.385	0.385
Negative Curve	-4	-0.049	0.370	2
Zero Curve	0	0	-1	-0.089
Monkey Saddle	-3.897	-0.053	0.405	1.997
Four-Point Saddle	-5.241	-0.061	0	2.396

Shapes	RMS		Absolute	
	κ_{min}	κ_{max}	κ_{min}	κ_{max}
Generic Graph	-	-	0.616	4.63
Positive Upward Curve	-	-	0.444	4
Positive Downward Curve	-	-	0.444	4
Negative Curve	0.805	4.899	0.864	5.657
Zero Curve	0.179	2	0.179	2
Monkey Saddle	0.944	4.871	1.045	5.614
Four-Point Saddle	0	5.768	$2.317x10^{-18}$	6.609

Table 2: Part II - Evaluating Curvatures on unknown Surfaces



(a) Gaussian and Mean Curvature



(b) Absolute and RMS Curvature

Figure 6: Curvature plots for Cylindrical surface

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Setting maxgc to +eps
Setting maxgc to +eps
Setting maxgc to +eps
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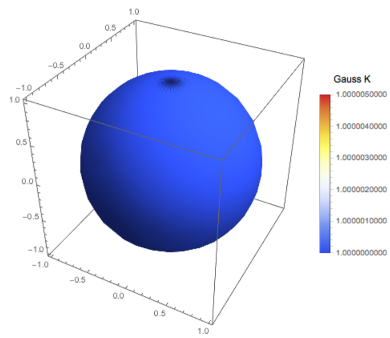
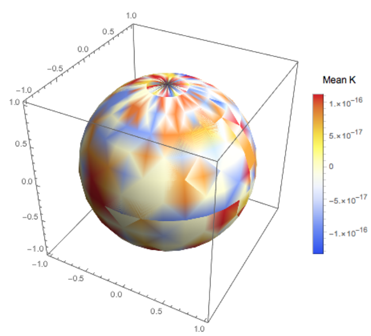
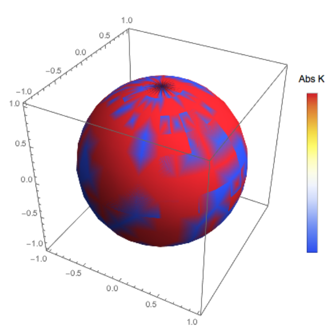


Color map of Gaussian curvature K
min K = 1.
max K = 1.

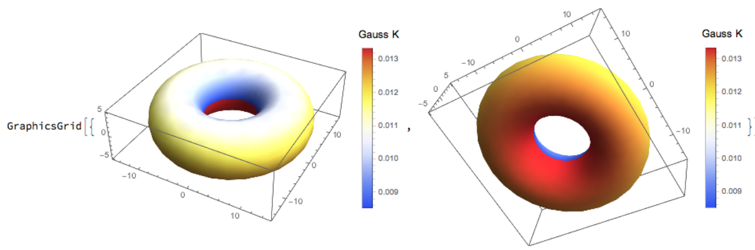


Color map of Mean curvature K
min K = -1.33562×10^{-16}
max K = 1.11556×10^{-16}



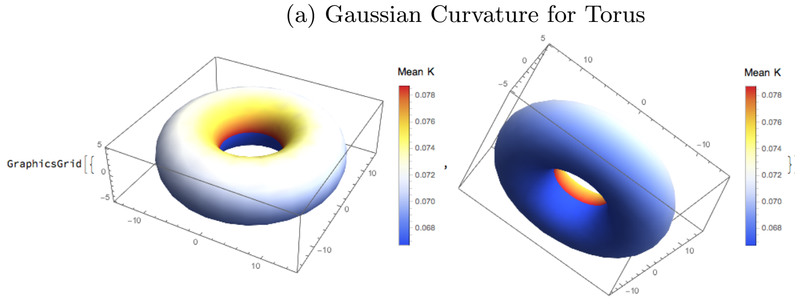
Color map of Abs curvature K
min K = 2.
max K = 2.

Figure 7: Curvature plots for Spherical surface



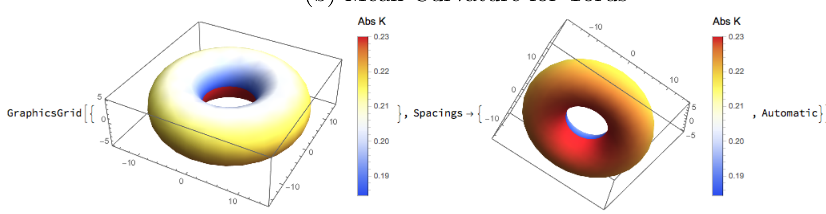
Color map of Gaussian curvature K
min K = 0.00850769
max K = 0.0133333

(a) Gaussian Curvature for Torus



Color map of Mean curvature K
min K = 0.0666667
max K = 0.0787308

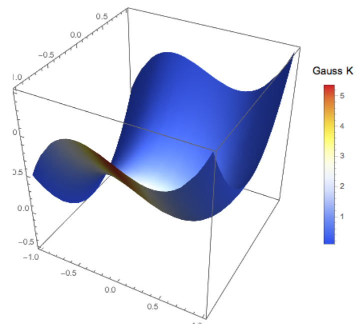
(b) Mean Curvature for Torus



Color map of Abs curvature K
min K = 0.184474
max K = 0.23094

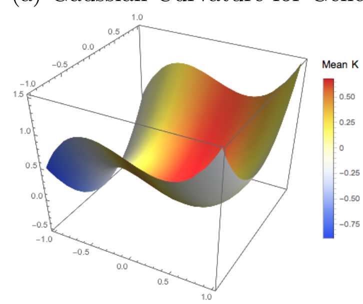
(c) Absolute Curvature for Torus

Figure 8: Curvature plots for Torus surface



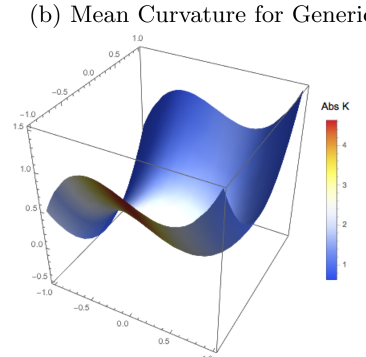

```
Color map of Gaussian curvature K over a generic graph
min K = 0.0948148
max K = 5.361
```

(a) Gaussian Curvature for Generic Graph




```
Color map of Mean curvature K over a generic graph
min K = -0.894427
max K = 0.67991
```

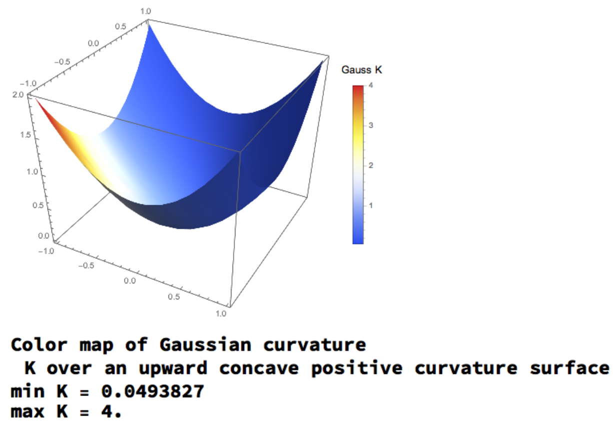
(b) Mean Curvature for Generic Graph



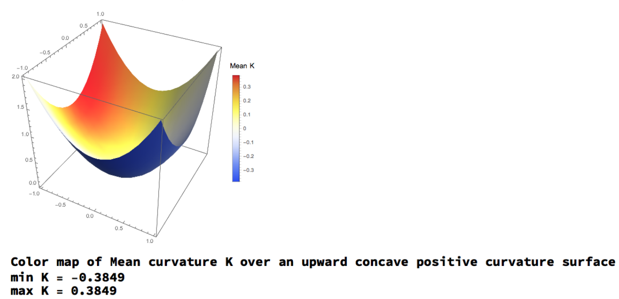

```
Color map of Absolute curvature K over a generic graph
min K = 0.61584
max K = 4.63077
```

(c) Absolute Curvature for Generic Graph

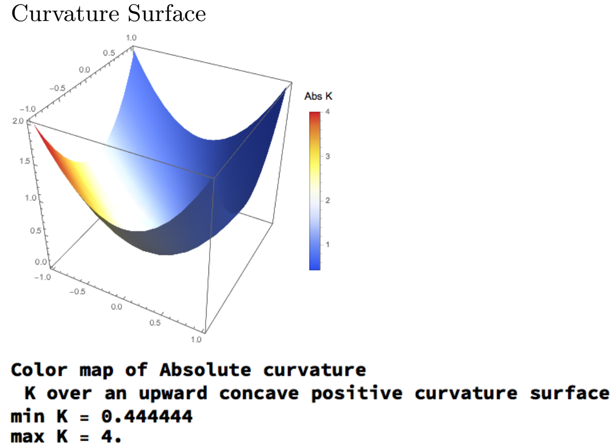
Figure 9: Curvature plots for Generic Graph



(a) Gaussian Curvature for Upward Positive Curvature Surface

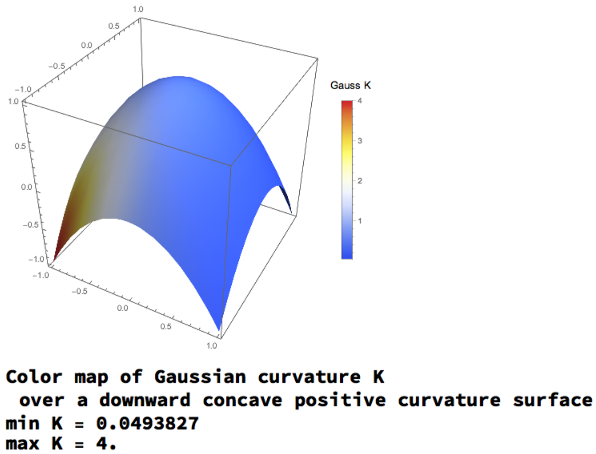


(b) Mean Curvature for Upward Positive Curvature Surface

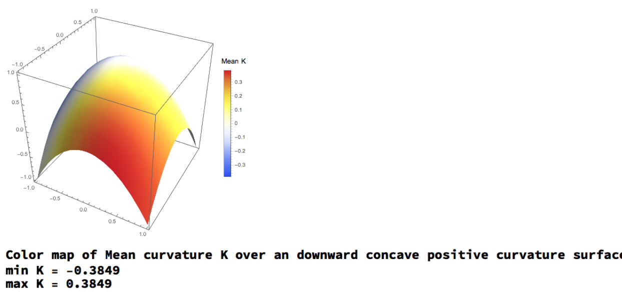


(c) Absolute Curvature for Upward Positive Curvature Surface

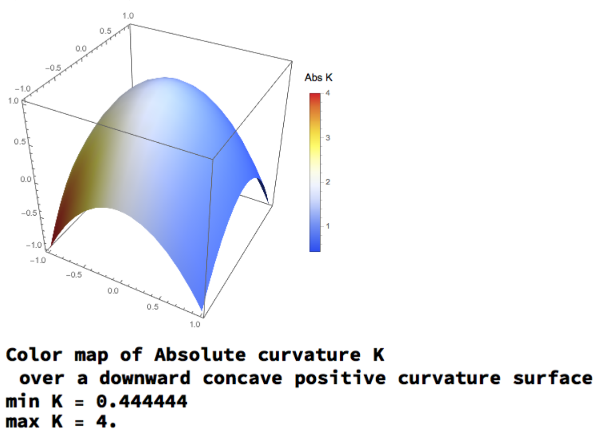
Figure 10: Curvature plots for Upward Positive Curvature Surface



(a) Gaussian Curvature for Downward Positive Curvature Surface

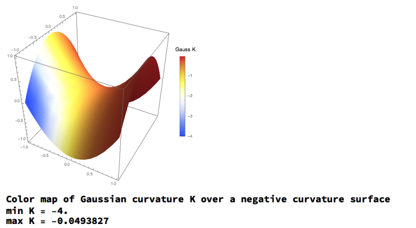


(b) Mean Curvature for Downward Positive Curvature Surface

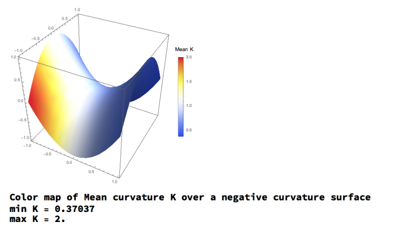


(c) Absolute Curvature for Downward Positive Curvature Surface

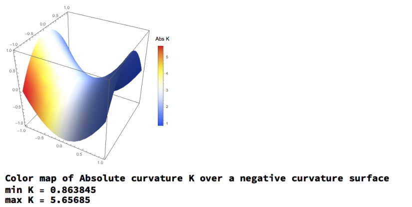
Figure 11: Curvature plots for Downward Positive Curvature Surface



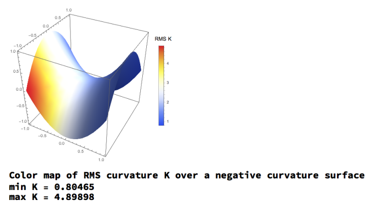
(a) Gaussian Curvature for Negative Curvature Surface



(b) Mean Curvature for Negative Curvature Surface

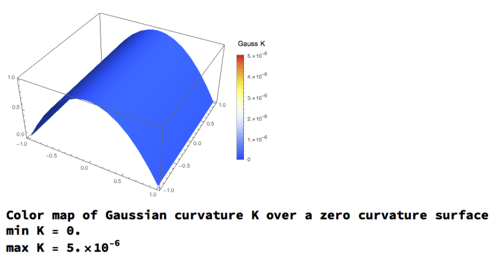


(c) Absolute Curvature for Negative Curvature Surface

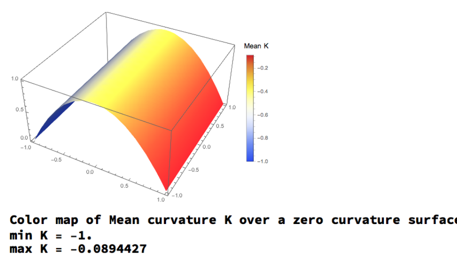


(d) RMS Curvature for Negative Curvature Surface

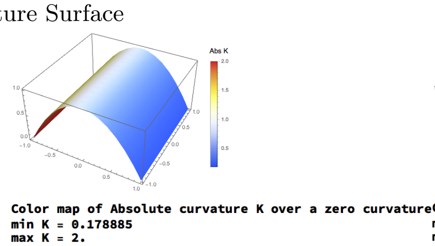
Figure 12: Curvature plots for Negative Curvature Surface



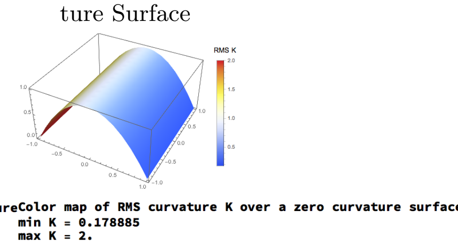
(a) Gaussian Curvature for Zero Curvature Surface



(b) Mean Curvature for Zero Curvature Surface

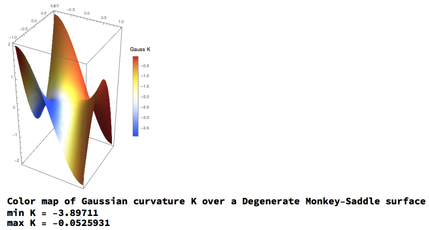


(c) Absolute Curvature for Zero Curvature Surface

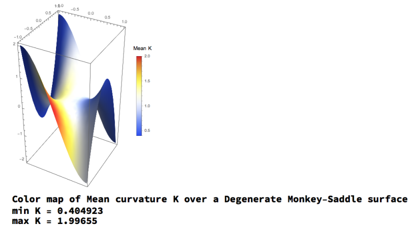


(d) RMS Curvature for Zero Curvature Surface

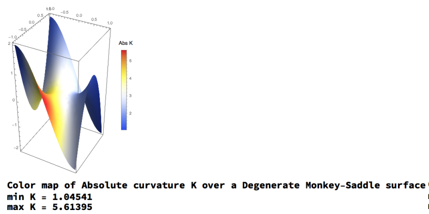
Figure 13: Curvature plots for Zero Curvature Surface



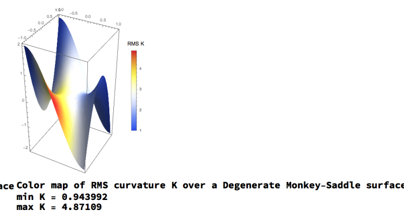
(a) Gaussian Curvature for Degenerate Monkey Saddle



(b) Mean Curvature for Degenerate Monkey Saddle

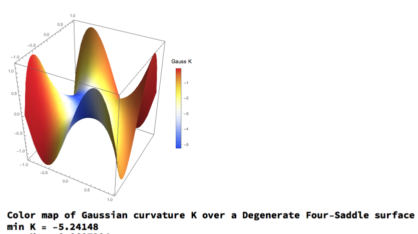


(c) Absolute Curvature for Degenerate Monkey Saddle

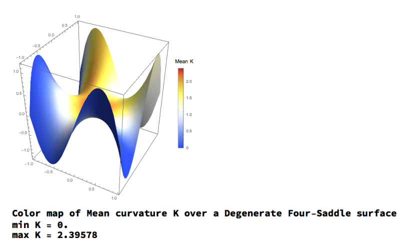


(d) RMS Curvature for Degenerate Monkey Saddle

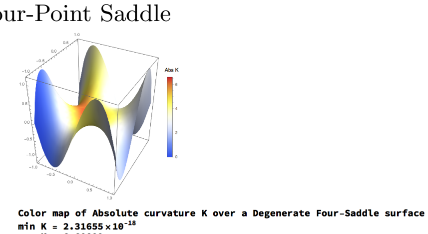
Figure 14: Curvature plots for Degenerate Monkey Saddle Surface



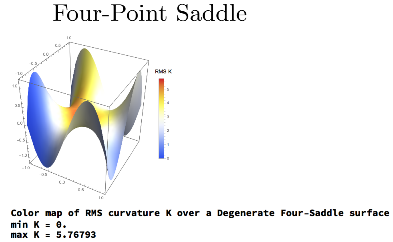
(a) Gaussian Curvature for Degenerate Four-Point Saddle



(b) Mean Curvature for Degenerate Four-Point Saddle



(c) Absolute Curvature for Degenerate Four-Point Saddle



(d) RMS Curvature for Degenerate Four-Point Saddle

Figure 15: Curvature plots for Degenerate Four-Point Saddle Surface

6 Conclusions

From the first part of the experiment, we can compare the results of the curvature values with the standard results and see that they match.

The Gaussian curvature for cylinder is constant and zero. For sphere and torus it is a positive constant value. Now for sphere it is a positive constant of 1 which is confirmed by the maximum and minimum curvature values both corresponding to 1. For torus, we do see a positive value, but its not a large value and the variation in min-max is also of only 0.0048 which is low. Hence, we can assume it to be almost constant for the whole surface implying an even distribution of curvature values which is true due to the symmetry element.

Also, the mean, RMS and the Absolute curvature of cylindrical is constant which should be as it is a zero curvature surface or what we termed earlier as a developable surface. We find the mean curvature of sphere and torus is very low establishing its symmetrical shape. We were not able to calculate the RMS curvature for sphere and torus because of some imaginary numbers being introduced in the calculations, but we could calculate the absolute curvature values and they are constant for sphere but with small variation for torus.

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